*Remember it is **how** you solve the tasks that counts – not just the specific results. You should writ, step by step, the method of solution/ideas in a strict and understandable way.*

Linear Algebra

Homework 1, Lectures 1-3

Vectors, complex numbers, polynomials

Review vectors; sum, dot product

$$\underline{u} = [x_1, x_2],\ \underline{v} = [y_1, y_2],\ \text{then}\ \ \underline{u}\cdot\underline{v} = |\underline{u}||\underline{v}|\cos\theta = x_1y_1 + x_2y_2$$

0. a) Find analytically and graphically the sum of vectors $\underline{u}$, $\underline{v}$, (the tails are in the origin of the coordinate system), their lengths and dot products

1. $\underline{u} = (3,1), \underline{v} = (3,1)$ 2. $\underline{u} = (4,2), \underline{v} = (0,1)$ 3. $\underline{u} = (-1,2), \underline{v} = (1,2)$

b) Find the the cosine of the angle between $\underline{v}$ and $\underline{w}$, which of the pairs of vectors are perpendicular (orthogonal)?

1. $\underline{u} = (3,-1), \underline{v} = (4,2)$ 2. $\underline{u} = (1,2), \underline{v} = (-2,1)$ 3. $\underline{u} = (0,1)\ , \underline{v} = (1,0)$

4. $\underline{u} = (1,1)\ , \underline{v} = (1,0)$

c). Determine all the vectors which are perpendicular (orthogonal) to the vector $\underline{v}$.

1. $\underline{u} = (1,2)$ 2. $\underline{u} = (\ 2,3)$ 3. $\underline{u} = (0,1)$.

Complex numbers

$$\boldsymbol{i^2 = -1 \qquad \sqrt{-1} = \{i, -i\}}$$

$$\boldsymbol{z = x + iy = r(\cos\alpha + i\sin\alpha) = r\,e^{i\,\alpha}}$$

$$\boldsymbol{\cos\alpha = \frac{x}{r}; \qquad \sin\alpha = \frac{y}{r};}$$

$$\boldsymbol{-\pi < Arg\,z \le \pi; \qquad \alpha = arg\,z = Arg\,z + 2k\,\pi.}$$

$$\boldsymbol{r = |z| = \sqrt{x^2 + y^2}\ ; \qquad \bar{z} = x - iy = re^{-i\alpha}}$$

$$\boldsymbol{z^n = r^n\,e^{in\alpha}}$$

$$\boldsymbol{\sqrt[n]{z} = \{z_0, z_1, z_2, \dots, z_k\};\ \ z_k = \sqrt[n]{r}\left(\cos\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right)\right)}$$

$$\boldsymbol{= \sqrt[n]{r}\ e^{i\left(\frac{\alpha}{n} + k\frac{2\pi}{n}\right)}}$$

1. Determine the following: Re [(2+5i)/(i-1)] ; conj (3+2i) ; |3-3i|

$a)\ Re[(2+5i)(3-4i)],\quad Im\,[(2+5i)(3-4i)]\quad$ $\boldsymbol{b})\ Re\left[\frac{2+5i}{i-1}\right],\quad Im\left[\frac{2+5i}{i-1}\right]$

$c)\ |3+5i-3+i|$ $d)\ \left|\frac{2+5i}{i-1}\right|$ $e)\ \overline{(-i+2)(4+2i)}$ $f)\ |a+bi-4+3i|$

$g)\ \left|\frac{i^7(1+i)^8}{\left(\sqrt{2}-i\sqrt{6}\right)^{12}}\right|$ $\boldsymbol{h})\ Im\left[\frac{i^7}{(2-2i)^4}\right]$

2. Find the absolute value $|z|$, the Real and Imaginary parts of z: $Re(z)$, $Im(z)$, for

$a)\ z=i^5+3i^7-3$ $b)\ z=(2+i^2+3i)(1-4i)$ $c)\ z=\ (1-i)^2$

$d)\ z=\frac{i^{10}}{(1-i)^{12}}$ $e)\ z=\frac{(1+i)(1+i)^2(1+i)^3\ldots(1+i)^{20}}{i^0+i^2+i^4+i^6+\cdots+i^{20}}$

3. Solve the following equations for $z\in C$, it is possible that there *are no solutions* or there are *more than one*. WAlpha: 2z+(1+i)conj(z)=1-3i

$a)\ z^2-z+1=0$ $b)\ z^2-2z+5=0$ $c)\ z^2+\sqrt{7}z+2=0$

$d)\ iz^2-z+2i=0$ $e)\ z^4+(1-i)\,z^2\ -i=0$ $f)\ 2z+(1+i)\overline{z}=1-3i$

$g)\ z^2=3+4i$ $h)\ (z+\overline{z})+2(z-\overline{z})=3+8i$ $i)\ \frac{z+1}{\overline{z}-1}=-1$

$j)\ \overline{z-i}=2z+1$ $k)\ 6+iz+z^2=0$ $l)\ -2z^2+6i^5\ -8\,i^{42}=0$

4. Sketch the following sets in the complex plane

$a)\ S=\left\{z\in C:\ Re[z]<Re\left[\frac{-3+2i}{2-i}\right]\right\}$

$b)\ S=\left\{z\in C:\ Im[z]>Im\left[\frac{-3+2i}{2-i}\right]\right\}$

$c)\ S=\left\{z\in C:\ Re\,[z]>Im\left[\frac{-3+2i}{2-i}\right]\right\}$

$d)\ S=\{z\in C:\ Re[(4-i)z]>Re\,[(-5+7i)(4+6i)]\}$

5. Calculate the argument arg (z) and the main argument $Arg(z)$, of z.

$a)\ Arg(1-i)\,;\ arg(1-i)\,,$ $b)\ Arg(\sqrt{3}+i);\ arg(\sqrt{3}+i),$

$c)\ Arg(\sqrt{2}-i\sqrt{6}),$ arg($\sqrt{2}-i\sqrt{6}$)

6. Plot the following points, find their polar form (i.e. trigonometric form) **and their exponential form***

$a)\ z=2i,$ $b)\ z=-3$ $\boldsymbol{c})\ z=-2+2i$ $d)\ z=-3i$ $e)\ z=-1-i$

$f)\ z=\ -\sqrt{3}+i$ $g)\ z=1+i\sqrt{3}$ $\boldsymbol{h})\ z=-4-i\,4\sqrt{3}$

$i)\ z=i^{33}+(1+i)^4$ $j)\ z=(-1+i)^8$

7. Sketch the following sets in the complex plane, mark the main points (wedges and circles)

$a)\ \ S=\{z\in C\colon |z+3-2i|\leq 2\ \}$ $\qquad b)\ \ S=\{z\in C\colon |z+3-2i|\leq |\sqrt{2}+2i|\ \}$

$c)\ \ S=\{z\in C\colon |\overline{z}+3-2i|\leq 2\ \}$ $\qquad d)\ \ S=\left\{z\in C\colon Arg(z)\leq \frac{\pi}{2}\right\}$

$e)\ \ S=\left\{z\in C\colon -\frac{\pi}{4}\leq Arg(z)\right\}$ $\qquad f)\ \ S=\left\{z\in C\colon -\frac{3\pi}{4}\leq arg(z)\leq \frac{\pi}{2}\ \right\}$

$g)\ \ S=\left\{z\in C\colon -\frac{3\pi}{4}\leq arg(\ \overline{z}\ )\leq \frac{\pi}{2}\ \right\}$ $\qquad h)\ S=\ \left\{z\in C\colon -\frac{3\pi}{4}\leq arg(z-2+i)\leq \frac{\pi}{2}\ \right\}$

$i)\ \ S=\{z\in C\colon\ \ Arg(i)\leq arg(z)\leq Arg(-1+i)\}$ $\qquad j)\ \ S=\{z\in C\colon |iz+3-2i|\leq 2\ \}$

$k)\ \ S=\left\{z\in C\colon\ -\frac{\pi}{2}\leq arg\big((i+1)\cdot z)\big)\leq \frac{\pi}{4}\right\}$ $\qquad l^{*})\ S=\left\{z\in C\colon\ \frac{\pi}{2}\leq arg(\ z^{3})\ \ \leq \frac{\pi}{2}\right\}$

$m)\ \ S=\left\{z\in C\colon\ 0\leq arg\left(\frac{z}{i}\right)\leq Arg(3+3i)\right\}$

$n)\quad S=\{z\in C\colon\ Arg(1-3i)\leq arg\, z\leq Arg(-2+5i)\}$

$m)\ S=\{\ \ z\in C\colon\ \mathrm{Im}[(2+i)(3+5i)]\geq \left|z-\overline{3+i}\right|\geq \left|\sqrt{5}+2i\ \right|$

$\qquad\qquad \wedge\ \mathrm{Arg}(3-i)\ \ \leq \mathrm{arg}(z)\leq \mathrm{Arg}\left[e^{\,i\frac{\pi}{2}}\right]\ \}$

8. Sketch the following sets in the complex plane, mark the main points (mixed regions)

$a)\ S=\left\{z\in C\colon\ 1\leq |z-1-i|<3,\ \ \ 0\leq Arg\, z\leq \frac{\pi}{2}\right\}$

$b)\ S=\{z\in C\colon Im[(1+2i)\cdot z-3i]<0\}$

$c)\ S=\{z\in C\colon |z-3+4i|<5\ ,Re\, z\geq 3,\ \ Im\, z<-3\}$ $\qquad d)\ \ S=\left\{z\in C\colon\ \overline{z+i}=z-1\right\}$

$e)\ S=\left\{z\in C\colon\ \ \frac{|3+2i|}{|z-3i-1|}\geq 2\ \right\}$ $\qquad f)\ \ S=\left\{z\in C\colon\ \frac{|z+3|}{|z-2i|}\geq 1\ \right\}$

$g)\ S=\{z\in C\colon |iz\ +1-i|<2\}$ $\qquad h)\ \ S=\left\{z\in C\colon \left|\overline{z-i+1}\right|<3\right\}$

$i)\ \ S=\{z\in C\colon\ |z-2i|+|z+2i|=4\}$ $\qquad j)\ \ S=\{z\in C\colon |\bar{z}\ +i|<2\}$

9. Find $Arg(z), |z|$ for the following complex numbers

$\boldsymbol{a})\ \ z=\left(e^{\frac{i\pi}{5}}\right)^{15}$ $\qquad \boldsymbol{b})\ \ z=(1+i)^{3}e^{\frac{i\pi}{4}}$ $\qquad \boldsymbol{c})\ z=\dfrac{(-3+3i)^{10}}{\left(e^{\frac{i\pi}{3}}\right)^{4}}$ $\qquad d)\ \ z=3\ i\ e^{\frac{i\pi}{4}}$

10. Let $z=-1+i$, write the following complex numbers in exponential form

$\boldsymbol{a})-z\qquad \boldsymbol{b})\ iz\qquad \boldsymbol{c})\ \frac{1}{z}$

11*. Let $z=2\left(\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right)$, write in exponential form

$a)-z,\qquad b)\ iz,\qquad c)\ \ 1/z,\qquad d)\ \ \overline{z},\qquad e)\big(1+i\sqrt{3}\big)\cdot z,\qquad f)\ z^{10}.$

12. First express the complex number z as in polar form , exponential form and in algebraical/canonical form $z = x + iy$.

$a)\ z = \left(\sqrt{3+i}\right)^{10}$ $\quad b)\ z = \dfrac{(1+i)^{10}}{(1-i)^8}$ $\quad c)\ z = \dfrac{(1+i)^{22}}{\left(1-i\sqrt{3}\right)^6}$

$d)\ z = \left(1-i\sqrt{3}\right)^6(-1-i)^4$ $\quad e)\ z = \dfrac{\left(e^{-\frac{i\pi}{7}}\right)^{49}}{\left(-\sqrt{2}+i\sqrt{6}\right)^{24}}$, $\quad f)\ z = \dfrac{i^{23}+i^{44}}{\left(-2-i2\sqrt{3}\right)^6}$

13*. Calculate the Cartesian coordinates of the point $Q(x,y)$ obtained by rotating point $P(2{,}3)$ by $60°$ around (0,0) (hint: use the multiplication of complex numbers).

14. Calculate and plot in the complex plane, the real and imaginary parts of the following ***numbers, remember there might be more than one value. Where possible find the algebraic values of the*** coordinates

$a)\ \sqrt{-4i}$ $\quad b)\ \sqrt[3]{i}$ $\quad c)\ \sqrt[5]{-1}$ $\quad d)\ \sqrt[3]{-1+i}$ $\quad e)\ \sqrt[4]{-81}$ $\quad \boldsymbol{f})\ \sqrt{2\sqrt{3}-2i}$

$\boldsymbol{g})\ \sqrt{5+12i}$ $\quad \boldsymbol{h})\ \sqrt{8+6i}$

15. Solve for z:

$a)\ \dfrac{z^4}{i^{14}+i^{17}} = 1,$ $\quad b)\ i^2 \cdot z^4 = i^6,$ $\quad \boldsymbol{c})\ z(1+i)^2 = 1,$ $\quad d)\ \dfrac{2z^3}{1-i} - 1 - i = 0,$

$e)\ \dfrac{i}{z^3} - \dfrac{1}{27i} = 0$ $\quad f)\ \dfrac{z^4}{i+1} = \sqrt{2}\, e^{i\frac{\pi}{4}}$

16. Let $z_1 = 3i + i^2,\ z_2 = \frac{2}{1-i}$. Plot these points in C.

$a)$ determine $z = z_1 + z_2,$

$b)$ determine $z = \sqrt[3]{z_1 + z_2}$,

$c)$ determine $z = z_1 \cdot z_2$

$d)$ determine $z = z_2^{44}$

***e)* give the geometric interpretation** of the above operations (sum, product, cubic root, power) and plot the results.

17*. Use the de Moivre'a Formula to determine the dependence of $\sin 2\alpha$ and $\cos 2\alpha$ on the functions $\sin\alpha$ and $\cos\alpha$.

18*. Use the exponential form to solve

$a)\ |z|^2 = iz^2,$ $\quad \boldsymbol{b})\ \dfrac{|z|^2\, z}{\bar{z}^3} = -1,$ $\quad z \neq 0$

19*. Write $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ in exponential form

a) calculate all the possible integer powers of z: $z^n, n \in I,\ \ I = Integers$

***b*)** the powers of z^i, where i is the imaginary unit $i^2 = -1$.

20 . Calculate the power e^i, where i is the imaginary unit $i^2 = -1$.

21. Find all the complex roots of the equations

a) $z^3 - z^2 + 3z + 5 = 0$ ***b***) $2z^3 + 4z^2 + 3z + 6 = 0$ ***c***) $z^3 + 2z^2 + z + 2 = 0$

d) $z^3 - \frac{7}{6}z^2 - \frac{3}{2}z - \frac{1}{3} = 0$

22. Let
a) $z = 2 + i$ be one of the roots of $z^4 - 2z^3 + 7z^2 - 30z + 50 = 0$ find all the other roots,

b) $z_1 = -i\sqrt{2}$, $z_2 = i$ be two of the roots of $z^6 - 2z^5 + 5z^4 - 6z^3 + 8z^2 - 4z + 4 = 0$ find all the other roots.

23. Write out a polynomial with real coefficients of the fourth degree which has the following roots: $z_1 = 1 - i$, $z_2 = 3i$.